Periodic Maximum Range Cruise with Singular Control

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Nomenclature

C_D = drag coefficient
C_L = lift coefficient
D = drag
E = energy
g = acceleration due to gravity
H = Hamiltonian
h = altitude
J = performance criterion
K = lift dependent drag factor ($C_D = C_{D_0} + KC_L^2$)
L = lift
m = mass
m_V = exponent denoting the dependence of thrust specific fuel consumption on speed
P = propulsive power, TV
S = reference area
$\bar{S}$ = switching function
T = thrust
V = speed
x = horizontal coordinate
y = state variables vector
γ = flight path angle
δ = throttle setting
λ = Lagrange multiplier
ν = specific fuel consumption related to propulsive power
ρ = atmospheric density
σ = specific fuel consumption related to thrust

Introduction

AIRCRAFT cruise classically consists of a steady-state flight at constant speed or altitude with a rectilinear flight path. In recent papers, however, it has been shown that this type of cruise is not generally optimal, and results have been presented for optimal aircraft cruise that show a periodic behavior, e.g., Refs. 1-7. The periodic cruise basically consists of a climbing flight at a high thrust setting followed by a sinking flight at a low or zero thrust setting. Accordingly, the control and state variables change in a periodic manner.

In the results presented so far for periodic optimal cruise for maximum range with time constraints excluded (i.e., free final time), the throttle appears to be purely bang-bang. This means that the throttle switches between its boundary values and does not take on values interior to its admissible set. Accordingly, the trajectory consists of periodically repeated cycles, each of which shows a maximum thrust phase followed by a minimum thrust phase. For singular control, thrust should be at intermediate settings for a finite interval of the optimal trajectory.

It is the purpose of this Note to show that singular arcs exist in optimal periodic cruise problems for minimizing fuel consumption per range, with a hypothetical propeller driven aircraft used as an example. Both mathematics and flight mechanics aspects concerning the existence of singular arcs will be considered.

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Problem Formulation

The optimal control problem consists of minimizing the fuel consumption for a given range. This problem can be formulated by introducing the performance criterion

$$J = m_f(x_{\text{cyc}})/x_{\text{cyc}} \tag{1}$$

where m_f is the fuel consumed and x_{cyc} is the (horizontal) length of one period of the trajectory. A period may be considered as a basic element of the whole trajectory.

The performance criterion is subject to the equations of motion, which may be written as

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{T - D - mg\sin\gamma}{mV\cos\gamma}, \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}x} = \frac{L - mg\cos\gamma}{mV^2\cos\gamma}$$

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \tan\gamma, \qquad \frac{\mathrm{d}m_f}{\mathrm{d}x} = \frac{\dot{m}_f}{V\cos\gamma} \tag{2}$$

The aerodynamic model for the lift and drag forces is

$$L = C_L(\rho/2)V^2S, \qquad D = C_D(\rho/2)V^2S \tag{3}$$

where

$$C_D = C_{D_0} + KC_L^2 \tag{4}$$

The models for thrust and fuel consumption may be expressed as

$$T(V,h,\delta) = \delta T_{\max}(V,h), \qquad \dot{m}_f = \delta\sigma(V)T_{\max}(V,h) \tag{5}$$

The mass of the aircraft can be considered constant for one period, because the fuel consumed during such an interval is small as compared with the total of the mass, i.e., $m_f(x_{\text{cyc}}) - m_f(0) \ll m$.

The following boundary conditions may be formulated

$$V(x_{\text{cyc}}) = V(0), \qquad \gamma(x_{\text{cyc}}) = \gamma(0)$$

$$h(x_{\text{cyc}}) = h(0), \qquad m_f(0) = 0 \tag{6}$$

Control variables are the lift coefficient C_L and the throttle setting δ, which are subject to the following inequality constraints

$$C_{L_{\min}} \le C_L \le C_{L_{\max}}, \qquad 0 \le \delta \le 1 \tag{7}$$

The atmospheric model used for describing air density and thrust dependence on altitude agrees with the *ICAO Standard Atmosphere*.[8]

The periodic control problem is to find the control histories C_L and δ, the initial states $[V(0), \gamma(0), h(0)]$, and the periodic cycle length x_{cyc} that minimize the performance criterion $J = m_f(x_{\text{cyc}})/x_{\text{cyc}}$ subject to the dynamic system described by Eq. (2), the boundary conditions given by Eq. (6), and the inequality constraints for the control variables, Eq. (7).

Optimality Conditions

Necessary conditions for optimality can be determined by applying the minimum principle. For this purpose, the Hamiltonian is defined as

$$H = \lambda_V \frac{T - D - mg\sin\gamma}{mV\cos\gamma} + \lambda_\gamma \frac{L - mg\cos\gamma}{mV^2\cos\gamma}$$

$$+ \lambda_h \tan\gamma + \lambda_f \frac{\sigma T}{V\cos\gamma} \tag{8}$$

where the Lagrange multipliers $\lambda = (\lambda_V, \lambda_\gamma, \lambda_h, \lambda_f)^T$ have been adjoined to the dynamic system of Eq. (2). The Lagrange multipliers are determined by

$$d\lambda/dx = -\partial H/\partial y \tag{9}$$

where $y = (V, \gamma, h, m_f)^T$. They are subject to the following boundary conditions:

$$\lambda_V(x_{\rm cyc}) = \lambda_V(0), \qquad \lambda_\gamma(x_{\rm cyc}) = \lambda_\gamma(0)$$

$$\lambda_h(x_{\rm cyc}) = \lambda_h(0), \qquad \lambda_f(x_{\rm cyc}) = 1/x_{\rm cyc} \tag{10}$$

The optimal controls C_L and δ are such that H is minimized. For this reason, C_L is determined either by $\partial H/\partial C_L = 0$ or by the constraining bounds of Eq. (7).

The system described by Eq. (2) is autonomous so that the Hamiltonian H is constant. Furthermore, because the cycle length $x_{\rm cyc}$ is considered free, the following relation holds:

$$H = m_f(x_{\rm cyc})/x_{\rm cyc}^2 \tag{11}$$

Singular Control Conditions

In regard to the throttle setting, H is linear in δ. Therefore, a bang-bang type of control and/or singular arcs can exist.

The throttle is bang-bang

$$\delta = 0 \quad \text{if} \quad \bar{S}(y,\lambda) > 0, \qquad \delta = 1 \quad \text{if} \quad \bar{S}(y,\lambda) < 0 \tag{12}$$

where $\bar{S}(y,\lambda) = \partial H/\partial\delta$ is the switching function.

A singular arc exists if $\bar{S}(y,\lambda) = 0$ for a finite interval of x. The order of the singular arc is $p = 1$. This is because $\partial\bar{S}^{(1)}/\partial\delta = 0$ and $\partial\bar{S}^{(2)}/\partial\delta \neq 0$ where $\bar{S}^{(i)} := d^i\bar{S}/dx^i$, $i = 1, 2$.

The relation $\bar{S}^{(2)}(y,\lambda,C_L,\delta) = 0$ can be used together with $\partial\bar{S}^{(2)}/\partial\delta \neq 0$ to give the singular control $\delta = \delta_{\rm sing}(y,\lambda,C_L)$. The switching structure of the optimal control is considered to be known. It may be expressed as

$$\delta = \delta_k(y,\lambda), \qquad x_k \le x \le x_{k+1}, \qquad k = 0,\ldots,s \tag{13}$$

where $0 = x_0 < x_1 < \cdots < x_s < x_{s+1} = x_{\rm cyc}$, $\delta_k \in \{0, 1, \delta_{\rm sing}\}$. The switching condition for a switching point between subarcs where the throttle is on its boundary is $\bar{S}[y(x_j),\lambda(x_j)] = 0$. For a singular subarc $x_j \le x \le x_{j+1}$, the switching condition is $\bar{S}^{(k)}[y(x_j),\lambda(x_j)] = 0$, $k = 0, 1$.

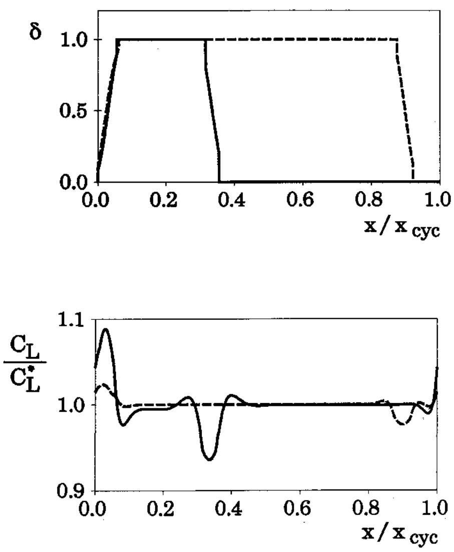

Fig. 1 Optimal period with singular arcs, throttle setting, and lift coefficient: —— $(L/D)_{\rm max} = 15$, $P_{\rm max}/(D_{\rm min}V_0^*) = 4.6$, $x_{\rm cyc} = 146$ km; ----- $(L/D)_{\rm max} = 15$, $P_{\rm max}/(D_{\rm min}V_0^*) = 1.2$, $x_{\rm cyc} = 142$ km.

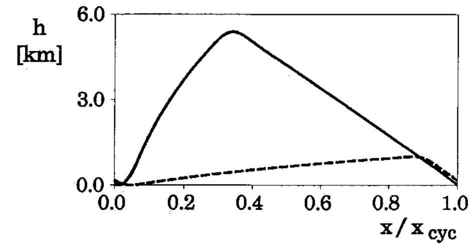

Fig. 2 Optimal period with singular arcs, altitude: —— $(L/D)_{\rm max} = 15$, $P_{\rm max}/(D_{\rm min}V_0^*) = 4.6$, $x_{\rm cyc} = 146$ km; ----- $(L/D)_{\rm max} = 15$, $P_{\rm max}/(D_{\rm min}V_0^*) = 1.2$, $x_{\rm cyc} = 142$ km.

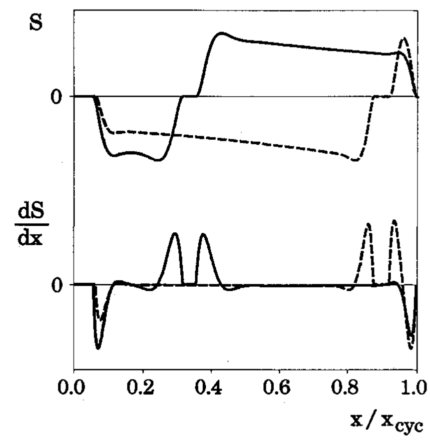

Fig. 3 Optimal period with singular arcs, switching function: —— $(L/D)_{\rm max} = 15$, $P_{\rm max}/(D_{\rm min}V_0^*) = 4.6$, $x_{\rm cyc} = 146$ km; ----- $(L/D)_{\rm max} = 15$, $P_{\rm max}/(D_{\rm min}V_0^*) = 1.2$, $x_{\rm cyc} = 142$ km.

Additional optimality conditions have been applied. They may be briefly described as the

1) Legendre Clebsch condition[9]: $H_{uu} \ge 0$,

2) Kelley condition[9]:

$$(-1)^k \frac{\partial}{\partial\delta}\left[\frac{d^{2k}\partial H}{dx^{2k}\partial\delta}\right] \ge 0$$

3) Goh condition[9]:

$$R = \begin{pmatrix} R_1 & R_2^T \\ R_2 & R_3 \end{pmatrix} \ge 0$$

with R_1, R_2, and R_3 accordingly defined, and

4) Robbins' equality condition[9]: $H_{u\lambda}H_{yu} - H_{uy}H_{\lambda u} = 0$ for all λ and y.

In the numerical investigation, an optimization program based on the method of multiple shooting was applied.[10-12] In these references, the numerical procedure is described in detail. The method applied provides results with high accuracy.

Results

Fuel consumption characteristics are a key factor concerning the existence of singular arcs in periodic optimal cruise. As will be discussed in more detail later on, singular arcs may exist in a pronounced manner when thrust specific fuel consumption shows a linear dependence on speed, i.e., $\sigma = \nu V$ where ν is a constant. This type of specific fuel consumption may be considered as representative for propeller driven aircraft, which show fuel flow characteristics proportional to propulsive power $\dot{m}_f = \nu P$, which may also be written as $\dot{m}_f = \nu P = \nu VT = \sigma T$. Accordingly, a vehicle with these characteristics is considered in the numerical investigation.

In Figs. 1 and 2, an optimal period for minimizing fuel consumption per range is presented. The throttle control (Fig. 1) shows a pronounced singular behavior for both transitions, during which it changes from one boundary value to the other. The singular control characteristic is further illustrated by the history of the switching function and its derivative, Fig. 3. The lift coefficient, which is the other control (Fig. 1), appears to be activated primarily during the singular arcs. The periodic optimal trajectory can be characterized as consisting of a climbing flight phase at maximum thrust setting followed by a sinking phase at zero thrust (Fig. 2). The singular thrust arcs may be considered as elements connecting these phases. It may be of interest to note that a state variable constraint ($h \geq h_{\min} = 0$) becomes active.

Figures 1 and 2 are also intended to illustrate the effect of maximum propulsive power available on singular control characteristics. As may be seen, the singular behavior in both cases considered is basically similar as far as the shape and the relative length of the singular arcs are concerned. This also holds for the switching function and its derivative. Differences more pronounced exist in regard to the lengths of the maximum and minimum thrust phases. In case of low propulsive power, the changes in lift coefficient and altitude are small so that the periodic behavior is rather close to steady-state flight. However, the singular characteristic remains similar to the high propulsive power case, which shows flight conditions of a more unsteady nature.

Flight Mechanics Considerations

A primary reason for the existence of singular arcs with periodic optimal cruise is the linear dependence of thrust specific fuel consumption on speed $\sigma = \nu V$. This can be shown by considering the speed influence on σ within the range possible for various powerplant systems.

Speed influence on specific fuel consumption for various types of powerplant systems may be described by introducing the following expression[13]:

$$\sigma = \sigma_0 (V_0 / V)^{m_V} \tag{14}$$

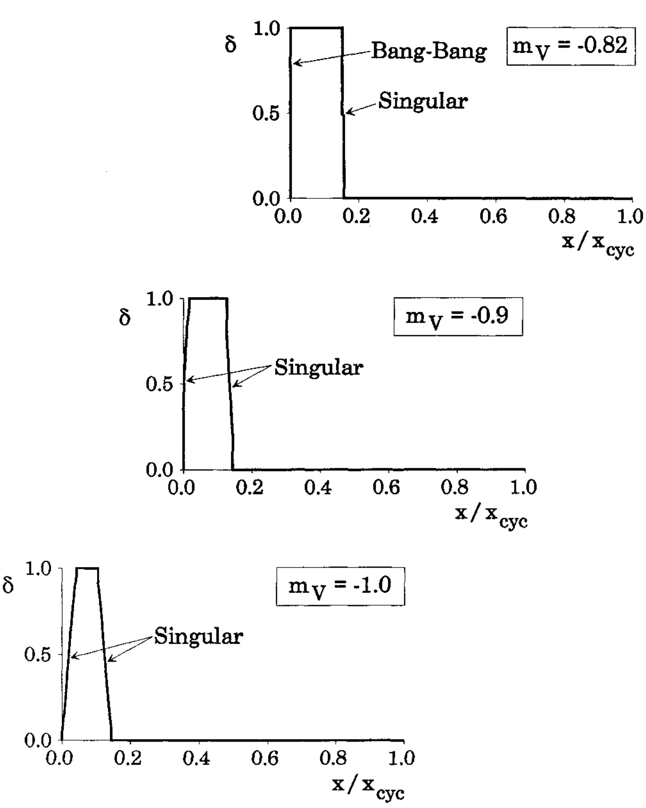

Fig. 4 Effect of m_V on singular control characteristics.

V_0 and σ_0 denote reference values, which can be chosen such that Eq. (14) fits realistic aircraft data for the airspeed and altitude range of interest (i.e., for the airspeed and altitude range covering the values of an optimal period). The exponent m_V is a quantity that can be used as representative for a specific powerplant system.[13] The following values may be regarded as typical:

$-m_V \approx 0,$	turbojet [equivalent to $\sigma \neq f(V)$, specific fuel consumption independent of speed]
$-m_V \approx -0.25,$	turbofan
$-m_V \approx -1.0,$	propeller (equivalent to $\sigma = \nu V$ with $\nu = \sigma_0 / V_0$ and denoting a fuel flow characteristic proportional to propulsive power $\dot{m}_f = \nu P$).

Accordingly, the following relation may be used to cover a wide range of different types of powerplant systems:

$$-1.0 \leq m_V \leq 0$$

Changing m_V within this range of practical interest has a significant effect on the control behavior concerning the existence of singular arcs. This is shown in Fig. 4. It is evident that deviations from $m_V = -1.0$ (representative for propeller driven aircraft) first reduce the length of the singular arcs and finally convert the singular thrust behavior into a pure bang-bang type of control. As may also be seen, the control behavior shown in the upper part of Fig. 4 is such that a singular arc only exists for the transition from maximum to minimum thrust, whereas the transition from minimum to maximum thrust shows bang-bang characteristics.

It may be of interest to note that period optimal cruise of an aircraft having a specific fuel consumption characteristic $\sigma = \nu V$ yields a decrease of the average of aerodynamic drag when compared with the minimum possible in steady-state cruise.[14]

Conclusions

For minimizing fuel consumption in aircraft range cruise, it is shown that periodic optimal trajectory solutions with singular control exist. Numerical results are presented for aircraft with a large and a small propulsive power. In both cases, a pronounced singular behavior exists. Fuel consumption characteristics are identified as a key factor. Singular arcs are favored when thrust specific fuel consumption is linearly dependent on speed. Such a characteristic is of practical significance because it may be existent with propeller driven aircraft.

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Performance of Higher Harmonic Control Algorithms for Helicopter Vibration Reduction

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I. Introduction

SIGNIFICANT vibration levels are encountered by helicopters because of variations in rotor blade aerodynamic loads with blade azimuth angle. These vibration levels reduce pilot effectiveness and passenger comfort and increase maintenance and operating costs. Thus, the control of helicopter rotors to reduce vibrations is of great interest.

Hooper[1] surveyed many major wind-tunnel and full-scale flight tests and determined that the primary cause of unsteady airloading above the second harmonic is the interaction of each rotor blade with the vortex of the preceding blade. The airloading of each blade is essentially periodic and can be decomposed as a sum of harmonics of the rotor frequency. For an N-bladed rotor, only vibrations at harmonics that are integer multiples of N times the rotor frequency are transmitted through the rotor mast, due to cancellation of the other harmonics by the N blades.

To reduce helicopter vibration, several methodologies known collectively as higher harmonic control (HHC) have been suggested. The basic idea behind HHC is to control rotor blades at harmonics of the rotor frequency so that unsteady airloads are canceled. One of the earliest HHC algorithms was suggested by Shaw,[2] with subsequent wind tunnel and flight tests.[3,4] In this approach, the control response matrix $\boldsymbol{T}$ is assumed to be known. The control response matrix relates the sine and cosine components of the N/rev swashplate inputs to the sine and cosine components of the N/rev response of the helicopter. At each step of the algorithm, a harmonic analysis of the measured quantity (either mast forces or accelerations at some fuselage location) is performed. The result is the sine and cosine components of the force (or acceleration) at the N/rev frequency. The vector of N/rev components is then multiplied by a decoupling matrix, which is the inverse of the control response matrix, to produce the change in commanded N/rev swashplate motion. Under the assumption that the control response of the helicopter to N/rev inputs is essentially quasisteady, this algorithm should eliminate N/rev vibrations in one step (that is, this method should produce deadbeat control).

Several authors[5-7] have proposed that stochastic adaptive controllers be used to account for parameter uncertainty and changing flight conditions. These methods are essentially derivative of Shaw's approach except that they include real-time identification of the control response matrix. The intent is to increase the robustness of the HHC algorithm to parameter uncertainty, changing flight conditions, and nonlinearities. However, this stochastic adaptive regulator implementation is much more complex than the fixed-gain controller advocated by Shaw. A review of these self-tuning regulators is provided by Johnson,[8] and some refinements are discussed by Davis.[9]

A different method, suggested by Gupta and DuVal[10] and DuVal et al.,[11] is the linear quadratic regulator with frequency-shaped cost functions, or LQR/FSCF. In this approach, the linear quadratic regulator is modified to allow for state and control penalties that are functions of frequency. By placing an infinite weight on the N/rev response, a controller is developed that is guaranteed to drive the N/rev response to zero.

A number of HHC algorithms have been the subject of wind-tunnel tests[3,12-15] and flight tests.[16-20] These tests have generally shown reductions in vibration levels of 25–90%. Shaw et al.[21] tested fixed-gain, gain-scheduled, and adaptive regulators on a dynamically scaled model of the three-bladed CH-47D Chinook rotor. This study showed that the fixed-gain regulator was 90% effective in suppressing three vibration components in almost all of the trimmed and quasisteady maneuvering envelope. Shaw also determined that a gain-scheduled controller provided no additional benefits over the fixed-gain controller. In testing two types of adaptive controllers, namely, global and local versions, the global controller was found to be unstable and the local controller was successful to the same extent as the fixed-gain regulator. Therefore, the fixed-gain regulator appears to be sufficient for eliminating most vibrations, and gain scheduling and/or adaptation of the $\boldsymbol{T}$ matrix is not of crucial importance.

In this Note, a framework is provided for the evaluation of HHC algorithm performance in terms of classical control theory. Single input/single output (SISO) characterizations of HHC algorithms are used to develop insight into the HHC problem. It is shown that HHC is fundamentally similar to the sinusoidal disturbance rejection techniques of classical control. By treating the periodic disturbance as a stochastic rather than a deterministic phenomenon, the performance of different HHC algorithms can be compared quantitatively. Furthermore, this framework allows the direct comparison of the discrete-time algorithms of Shaw and others to continuous-time algorithms, such as those of Gupta and DuVal.[10] Finally, this framework allows the investigation of the effects of model uncertainty due to parameter uncertainty and changing flight conditions.

II. Quasisteady Approach

Many authors, including Shaw et al.,[3,4,13,21,22] McCloud,[23-25] Wood et al.,[19] and Molusis,[5,6] represent the dynamic response of a helicopter to control inputs at the N/rev frequency by a constant matrix $\boldsymbol{T}$ that relates the Fourier coefficients of the N/rev harmonics of the swashplate commands to the N/rev harmonics of the vibration. This approach eliminates the need for a detailed model of the periodic helicopter dynamics, but requires that the controller bandwidth be low enough that the dynamics of the helicopter can be treated as quasisteady. In this section it will be demonstrated that the algorithm proposed by Shaw (and similar algorithms) behave similarly to the

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